Consumption and Asset Pricing

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References:

- Williamson’s lecture notes (2006) ch5 and ch 6
- Further references:
  - Asset pricing: Cochrane (2005)
Background Knowledge

- Expected Utility Theory
  - Risk aversion
- Stochastic dynamic programming
  - Brock and Mirman (1972)
Expected Utility Theory 1

Consumer’s preferences

- Deterministic world: ranking in consumption bundles
- Uncertainty: ranking in lotteries

**Example:** A world with a single consumption good $c$

Lottery 1: \[
\begin{cases}
  c_1^1, & \text{with prob. } p_1 \\
  c_2^1, & \text{with prob. } 1 - p_1
\end{cases}
\]

Lottery 2: \[
\begin{cases}
  c_1^2, & \text{with prob. } p_2 \\
  c_2^2, & \text{with prob. } 1 - p_2
\end{cases}
\]

- Expected utility from lottery $i$, $i = 1, 2$, is
  \[p_i u\left(c_i^1\right) + \left(1 - p_i\right) u\left(c_i^2\right)\]

- The consumer ranks lottery 1 and 2 according to
  \[p_i u\left(c_1^1\right) + \left(1 - p_i\right) u\left(c_1^2\right) \geq p_i u\left(c_2^1\right) + \left(1 - p_i\right) u\left(c_2^2\right)\]
Expected Utility Theory 2

- Risk aversion
  - Many aspects of observed behavior toward risk is consistent with risk aversion
  - If the utility function is strictly concave, then the consumer is risk averse
  - Jensen’s inequality

\[ E[u(c)] \leq u(E[c]) \]
Expected Utility Theory 3
Expected Utility Theory 4

- Anomalies in observed behavior towards risk
  - The Allais Paradox

- Two measures of risk aversion
  - **Absolute risk aversion**
    
    \[ ARA(c) = -\frac{u''(c)}{u'(c)} \]

    - Example: \( u(c) = -e^{-\alpha c} \).

  - **Relative risk aversion**
    
    \[ RRA(c) = -\frac{u''(c) c}{u'(c)} \]

    - Example: \( u(c) = \frac{c^{1-\sigma}-1}{1-\sigma} \).
Stochastic Optimal Growth Model

- Brock and Mirman (1972): stochastic optimal growth model
- The representative consumer's preferences

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \]

where \( 0 < \beta < 1 \) and \( u(\cdot) \) strictly increasing, strictly concave and twice differentiable.
- \( E_0 \): expectation operator conditional on information at \( t = 0 \).
- Production technology

\[ y_t = z_tF(k_t, n_t) \]

- \( z_t \): random technology disturbance
- \( \{z_t\}_{t=0}^{\infty} \): a sequence of i.i.d. random variables drawn from \( G(z) \)
- Law of motion for capital

\[ k_{t+1} = i_t + (1 - \delta) k_t, \quad 0 < \delta < 1 \]
- Resource constraint: \( c_t + i_t = y_t \)
Competitive equilibrium: Two approaches

1. Arrow and Debreu (Arrow (1983) and Debreu (1983))
   - At $t = 0$, market for contingent claims is opened and the representative consumer and the representative firm trade
   - State-contingent-commodities/claims: a promise to deliver a specified number of units of a particular object (labor or capital services) at a particular date $T$ conditional on $\{z_0, z_1, z_2, ..., z_T\}$
   - All markets clear
Arrow-Debreu Eqm vs Rational Expectation Eqm 2

2. Spot market trading with rational expectations (Muth (1960))

1. • At each date the consumer rents capital and sells labor at market price
   • The consumer makes optimal saving decision based on his/her beliefs about the prob. distribution of future prices
   • The markets clear at every date $t$ for every possible realization of the random shock $\{z_0, z_1, z_2, ..., z_t\}$
   • All expectations are rational: beliefs of the prob. distribution $=$ actual prob. distribution

• Both equilibria are Pareto Optimal (but not true in models with heterogenous agents)
Social Planner’s Problem

- Social planner’s problem

$$\max_{\{c_t, k_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. $$c_t + k_{t+1} = z_t f(k_t) + (1 - \delta) k_t$$

where $$f(k) \equiv F(K, 1)$$

- The Bellman equation:

$$v(k_t, z_t) = \max [u(c_t) + \beta E_t v(k_{t+1}, z_{t+1})]$$

s.t. $$c_t + k_{t+1} = z_t f(k_t) + (1 - \delta) k_t$$

- Note that $$c_t$$ is known but $$c_{t+i}, i = 1, 2, 3, \ldots$$, is unknown (uncertain)

- Goal: solve for $$v(k, z)$$ and the optimal decision rule

$$k_{t+1} = g(k_t, z_t) \text{ and } c_t = z_t f(k_t) + (1 - \delta) k_t - k_{t+1}$$
Example

- \( u(c_t) = \ln c_t, f(k_t) = k_t^\alpha n_t ^{1-\alpha}, 0 < \alpha < 1, y_t = z_t F(k_t, n_t), \delta = 1 \) and \( E[\ln z_t] = \mu \)

- **Guess and verify**
  - Guess that the value function takes the form
    \[ v(k_t, z_t) = A + B \ln k_t + D \ln z_t \]
  - It can be solved that
    \[ k_{t+1} = \alpha \beta z_t k_t^\alpha \]
    \[ c_t = (1 - \alpha \beta) z_t k_t^\alpha \]

- The economy will NOT converge to a steady state
  - Technology disturbances will cause persistent fluctuations in output, consumption and investment
  - Stochastic steady state

- Problems with the model
  - \( \text{Var}(\ln k_{t+1}) = \text{Var}(\ln y_t) = \text{Var}(\ln c_t) : \) not the case in the data
Asset Pricing Model

- Lucas (1978)
  - Treat consumption and output as exogenous and asset prices as endogenous
  - Also called as the ICAPM (intertemporal capital asset pricing model) or consumption-based capital asset pricing model
The Economy 1

- Representative agent

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \]

- 0 < \beta < 1, \( u(\cdot) \) strictly increasing, strictly concave, twice differentiable

- Output
  - Exists \( n \) productive units (fruit trees), denote the productive unit by \( i, i = 1, \ldots, n \)
  - \( y_{it} \): quantity of output produced/yielded by production unit \( i \) at time \( t \), a random variable

- Equilibrium:

\[ c_t = \sum_{i=1}^{n} y_{it} \]
The Economy 2

• Asset holding
  
  • \( p_{it} \): price of tree \( i \) at time \( t \)
  • \( z_{it} \): shares of tree held at time \( t \)

Goal: determine the prices of the trees

  • Shares are traded in competitive market
  • Endowment: \( z_i^0 = 1, i =, ..., n \)
  • The fruits/output on each tree is proportionally distributed to their share holders according to their share holding
  • After the distributing of fruits, the shares are traded again

• Budget constraint

\[
\sum_{i=1}^{n} p_{it} z_{i,t+1} + c_t = \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) \tag{1}
\]

for \( t = 0, 1, 2, \ldots \)
The Bellman equation

\[ v(z_t, p_t, y_t) = \max_{c_t, z_{t+1}} \left[ u(c_t) + \beta E_t v(z_{t+1}, p_{t+1}, y_{t+1}) \right] \]

\[ s.t. \quad c_t = \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) - \sum_{i=1}^{n} p_{it} z_{i, t+1} \]

Rewrite the Bellman equation as

\[ v(z_t, p_t, y_t) = \max_{c_t, z_{t+1}} \left[ u \left( \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) - \sum_{i=1}^{n} p_{it} z_{i, t+1} \right) + \beta E_t v(z_{t+1}, p_{t+1}, y_{t+1}) \right] \]

FOC and envelope theorem

\[-p_{it} u' \left( \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) - \sum_{i=1}^{n} p_{it} z_{i, t+1} \right) + \beta E_t \frac{\partial v}{\partial z_{i, t+1}} = 0 \]

\[ \frac{\partial v}{\partial z_{i, t+1}} = (p_{it} + y_{it}) u' \left( \sum_{i=1}^{n} z_{it} (p_{it} + y_{it}) - \sum_{i=1}^{n} p_{it} z_{i, t+1} \right) \]

for \( i = 1, \ldots, n \).
• *The basic formula for asset pricing*

\[
p_{it} = E_t \left[ (p_{i,t+1} + y_{i,t+1}) \hat\beta u'(c_{t+1}) \right. \left. \frac{u'(c_t)}{u'(c_t)} \right]
\]
Optimization 3

- Let

\[ \pi_{it} = \frac{p_{i,t+1} + y_{i,t+1}}{p_{i,t}} \quad \text{(gross return)} \]

\[ m_t = \frac{\beta u'(c_{t+1})}{u'(c_t)} \]

- Equation (2) can be rewritten as

\[ E_t(\pi_{it} m_t) = 1 \]

- Use \( \text{cov}(X, Y) = E(XY) - E(X)E(Y) \)

\[ \text{cov}(\pi_{it}, m_t) + E_t(\pi_{it}) E_t(m_t) = 1 \]

- What is a good asset?

  - A good asset is one that pays you well when your consumption is low.
Optimization 4

- Apply the law of iterated expectations

\[ E_t [E_{t+s} x_{t+s'}] = E_t x_{t+s'}, \quad s' \geq 0, \quad s \geq 0 \]

- The price of tree \( i \) at time \( t \) can be rewritten as

\[
p_{it} = E_t \left[ (p_{i,t+1} + y_{i,t+1}) \frac{\beta u'(c_{t+1})}{u'(c_t)} \right]
\]

\[
= E_t \left[ \frac{\beta u'(c_{t+1})}{u'(c_t)} y_{i,t+1} + \frac{\beta^2 u'(c_{t+2})}{u'(c_t)} y_{i,t+2} + \frac{\beta^3 u'(c_{t+3})}{u'(c_t)} y_{i,t+3} + \cdots \right]
\]

\[
= E_t \left[ \sum_{s=t+1}^{\infty} \frac{\beta^{s-t} u'(c_s)}{u'(c_t)} y_{is} \right]
\]

- That is, the current price of any asset is the expected discounted value of future dividends, where the discount factor is the IMRS
Example 1

- Assume that $y_{it}$ is i.i.d., and hence $p_{it}$ is i.i.d.

$$E_t \left[ (p_{i,t+1} + y_{i,t+1}) u' \left( \sum_{i=1}^{n} y_{i,t+1} \right) \right] = A_i$$

for $i = 1, \ldots, n$, $A_i > 0$ is a constant.

- From the basic asset-pricing formula

$$p_{it} = \frac{\beta A_i}{u' \left( \sum_{i=1}^{n} y_{i,t} \right)}$$

- If $\sum_{i=1}^{n} y_{i,t}$ low $\rightarrow$ $u'$ high $\rightarrow$ $p_{it}$ low

- If $\sum_{i=1}^{n} y_{i,t}$ high $\rightarrow$ $u'$ low $\rightarrow$ $p_{it}$ high
Example 2: Risk Neutral Agent

- Assume that \( u(c) = c \)
- From the basic asset-pricing formula

\[
p_{it} = \beta E_t \left[ p_{i,t+1} + y_{i,t+1} \right]
\]

\[
\implies E_t \left[ \frac{p_{i,t+1} + y_{i,t+1} - p_{it}}{p_{it}} \right] = \frac{1}{\beta} - 1
\]

- Familiar formula in finance: only hold when people are risk neutral
- \( p_{it} \) can be rewritten as

\[
p_{it} = E_t \sum_{s=t+1}^{\infty} \beta^{s-t} y_{is}
\]
Example 3

- Assume that \( u(c) = \ln(c) \), \( n = 1 \) and

\[
y_t = \begin{cases} 
  y_1 & \text{w.p. } \pi \\
  y_2 & \text{w.p. } 1 - \pi
\end{cases}
\]

, \( y_1 > y_2 \), \( y_t \) is i.i.d.

- Let \( p_i \) denote the price of a share when \( y_t = y_i \) for \( i = 1, 2 \).

\[
p_1 = \beta \left[ \pi \frac{y_1}{y_1} (p_1 + y_1) + (1 - \pi) \frac{y_1}{y_2} (p_2 + y_2) \right]
\]

\[
p_{1t} = \beta \left[ \pi \frac{y_2}{y_1} (p_1 + y_1) + (1 - \pi) \frac{y_2}{y_2} (p_2 + y_2) \right]
\]

- Can solve for

\[
p_1 = \frac{\beta y_1}{1 - \beta}
\]

\[
p_2 = \frac{\beta y_2}{1 - \beta}
\]

since \( y_1 > y_2 \), it follows that \( p_1 > p_2 \).
The Equity Premium Puzzle 1

- The average rate of return on equity is approximately 6% higher than the average rate of return on risk-free debt
- Mehra and Prescott (1985) showed that to generate such a big equity premium in Lucas’ asset pricing model, the implied IES of consumption must be very large, lying outside of the range of estimates for IES in empirical work

- 2 assets: a risk-free asset and a risky asset
- Risky-asset
  \[ \Pr [y_{t+1} = y_j | y_t = y_i] = \pi_{ij} \]
  where \( \pi_{ij} = \rho, \; 0 < \rho < 1 \)
- Derive \( p_i, q_i, \; i = 1, 2 \) when \( y_t = y_i \) for \( i = 1, 2 \) and derive \( R_1, R_2, r_1, r_2 \)
- The average equity premium
  \[ e (\beta, \sigma, \rho, y_1, y_2) = \frac{1}{2} (R_1 - r_1) + \frac{1}{2} (R_2 - r_2) \approx 0.06 \]
The Equity Premium Puzzle 2

- Two explanations \( u(c) = c^{1-\sigma} / (1 - \sigma) \)
  - 1. The higher the \( \sigma \) is, the lower the IES is, and the greater is the tendency of the representative consumer to smooth consumption over time. Or, the higher the \( \sigma \) is, the less willing the agents are to save for future consumption. Hence, to induce the agent to save more, have to give more compensation \( (r_t \uparrow) \)
  - 2. \( \sigma \) also measures for risk aversion. The higher is \( \sigma \) the larger is the expected return on equity, as agents must be compensated more for bearing risk
- Fitting the model into data: not enough variability in aggregate consumption to produce a large enough risk premium, given plausible levels of risk aversion